

A comparison of Vlasov with drift kinetic and gyrokinetic theories

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Abstract

A kinetic consideration of an axisymmetric equilibrium with vanishing electric field near the magnetic axis shows that ∇f should not vanish on axis within the framework of Vlasov theory while it can either vanish or not in the framework of both a drift kinetic and a gyrokinetic theories (f is either the pertinent particle or the guiding center distribution function). This different behavior, relating to the reduction of phase space which leads to the loss of a Vlasov constant of motion, may result in the construction of different currents in the reduced phase space than the Vlasov ones. This conclusion is indicative of some limitation on the implications of reduced kinetic theories in particular as concerns the physics of energetic particles in the central region of magnetically confined plasmas.

There are important phenomena in plasma physics as microinstabilities which can not be described in the framework of a macroscopic theory as magnetohydrodynamics but require the employment of kinetic theory. For the high temperature fusion plasmas an appropriate approximate kinetic theory is based on the Vlasov equation which the Boltzmann/Fokker-Planck equation reduces to when the collision term is neglected. To solve self consistently the set of the Maxwell-Vlasov equations, however, is a tough problem related to the fact that the complete set of constants of motion is missing. For example, in the case of an axisymmetric equilibrium only the energy and the canonical momentum conjugate to the toroidal coordinate are known conserved quantities out of the four potential constants of motions. Because of the two missing constants of motion it is not possible to construct equilibria with macroscopic poloidal velocities, although phenomenologically sheared poloidal velocities play an important role in the transition to improved confinement regimes of magnetically confined plasmas, e.g. the L-H transition. In addition, the temporary and probably future computational efficiency puts a limitation on numerical solutions of the Maxwell-Vlasov equations. For this reason approximate kinetic theories in a reduced phase space, as the drift kinetic [1]-[6] and gyrokinetic ones [7]-[15] have been developed and applied to numerous simulations, e.g. on turbulence in connection with the creation of zonal flows. In both theories the reduced phase space is five dimensional with three spatial components associated with the guiding center position, \mathbf{R} (or the gyrocenter position in the framework of gyrokinetic theory), and a velocity component parallel to the magnetic field, v_{\parallel} ; also, the two components of the perpendicular particle velocity are approximated after a gyroangle averaging with the magnetic moment which is treated as an adiabatic invariant. A related underlying assumption for both reduced theories is that the ratio ϵ of the gyroperiod to the macroscopic time scale is small. In the drift kinetic theory ϵ is the same as the ratio of the gyroradius to macroscopic scale length while in the gyrokinetic theory small spatial variations are permitted but, for example, the amplitudes of the fluctuations to the background fields is equal to ϵ [10]. It may be noted here that the reduced-phase-space kinetic theories are developed via expansions in ϵ the convergence of which is not guaranteed.

Because of the reduction of the phase space some information of the particle motion is missing. This gives rise to the question: is the missing information important? In the present note we address this question by making a comparison of Vlasov with drift kinetic and gyrokinetic theories near the magnetic axis of an axisymmetric magnetically confined plasma

with vanishing electric fields. Motivation was a previous study [16] in which by considering this equilibrium in the framework of Vlasov theory we found the following new constant of motion: $C_1 = v_z + I \ln |v_\phi|$, where v_ϕ is the toroidal particle velocity, v_z the velocity component parallel to the axis of symmetry and $\mathbf{B}_\phi = I/r \mathbf{b}_\phi$ the toroidal magnetic field near axis (r, ϕ, z are cylindrical coordinates). For the sake of notation simplicity and without loss of generality here we will consider only ions and employ convenient units by setting $m = q = c = 1$ where m and q are the ion mass and charge and c is the velocity of light. Since phenomenologically the density gradients vanish on the magnetic axis it was a surprising conclusion of Ref. [16] that $\nabla f \neq 0$ must hold on axis, where f is the particle distribution function. The reason is that if one assumes $\nabla f = 0$ thereon it is not possible to obtain one of the known constants of motion, i.e. the canonical toroidal momentum $C_2 = r v_\phi$. Note also that because of the absolute value of v_ϕ in C_1 , distribution functions depending on C_1 and the energy $C_3 = 1/2[v_\phi^2 + v_z^2 + v_r^2]$ can not create currents parallel to the magnetic field.

The form of C_1 relates crucially to the toroidicity because in the straight case for which ϕ changes to z , z to y and r to x (where x, y, z are Cartesian coordinates) for a z -independent equilibrium with straight magnetic axis parallel to z and arbitrary cross sectional shape the respective constant of motion becomes $C_1 = v_z$. Thus, in this case distribution functions depending on C_1 and the energy can produce parallel currents. It may also be noted that in the straight case for $\nabla f \neq 0$ on axis one can obtain the complete set of four constants of motion near axis: $C_1 = v_z$, $C_2 = v_x - B_z y$, $C_3 = v_y + B_z x$ and $C_4 = 1/2(v_x^2 + v_y^2 + v_z^2)$ where B_z is the “toroidal” magnetic field on axis [16]. Therefore, distribution functions of the form $f(C_2, C_3, C_4)$ can lead to purely “poloidal” velocities near axis irrespective of the cross sectional shape in consistence with magnetohydrodynamics [18]. Unlikely, toroidal magnetohydrodynamic equilibria with purely poloidal velocities are not possible [19]. Whether this conclusion of nonexistence persists in the framework of Vlasov theory is an open question relating to the two unknown constants of motion. The above comparison shows that the Vlasov theory well distinguishes equilibria with circular and straight magnetic axes.

In the present note we examine the same equilibrium near axis in the framework of drift kinetic and gyrokinetic theories on an individual basis. Though the drift kinetic equations of Ref. [3] and the gyrokinetic equations of Ref. [7] will be employed we claim that the conclusions do not rely on the particular forms of the reduced kinetic equations.

Drift kinetic theory

The drift kinetic theory established in Ref. [3] is based on the Littlejohn's Lagrangian for the guiding center motion [17] extended to include the polarization drift in such a way that local conservation of energy is guaranteed. The drift kinetic equation for the guiding center distribution function $f(\mathbf{R}, v_{\parallel}, \mu, t)$ (with $\dot{\mu} = 0$) acquires the form

$$\frac{\partial f}{\partial t} + \mathbf{v} \cdot \nabla f + \dot{v}_{\parallel} \frac{\partial f}{\partial v_{\parallel}} = 0. \quad (1)$$

The guiding center velocity, \mathbf{v} , and the “acceleration” parallel to the magnetic field, \dot{v}_{\parallel} , can be expressed in a concise form by introducing the modified potentials, \mathbf{A}^* and Φ^* , and the respective modified electric and magnetic fields, \mathbf{E}^* and \mathbf{B}^* , as [4]:

$$\mathbf{A}^* = \mathbf{A} + v_{\parallel} \mathbf{b} + \mathbf{v}_E, \quad (2)$$

$$\Phi^* = \Phi + \mu B + \frac{1}{2}(v_{\parallel}^2 + v_E^2), \quad (3)$$

$$\mathbf{v}_E = \frac{\mathbf{E} \times \mathbf{B}}{B^2}, \quad (4)$$

$$\mathbf{E}^* = -\frac{\partial \mathbf{A}^*}{\partial t} - \frac{\partial \Phi^*}{\partial \mathbf{R}}, \quad (5)$$

$$\mathbf{B}^* = \nabla \times \mathbf{A}^*, \quad (6)$$

where Φ and \mathbf{A} are the usual electromagnetic scalar and vector potentials and $\mathbf{b} = \mathbf{B}/B$. The quantities \mathbf{v} and \dot{v}_{\parallel} are then given by

$$\mathbf{v} = \mathbf{v}_g = v_{\parallel} \frac{\mathbf{B}^*}{B_{\parallel}^*} + \frac{\mathbf{E}^* \times \mathbf{b}}{B_{\parallel}^*}, \quad (7)$$

$$\dot{v}_{\parallel} = \frac{\mathbf{E}^* \cdot \mathbf{B}^*}{B_{\parallel}^*} = \frac{1}{v_{\parallel}} \mathbf{v}_g \cdot \mathbf{E}^*, \quad (8)$$

where

$$B_{\parallel}^* = \mathbf{B}^* \cdot \mathbf{b} = B + v_{\parallel} \mathbf{b} \cdot \nabla \times \mathbf{b} + \mathbf{b} \cdot \nabla \times \mathbf{v}_E. \quad (9)$$

Explicit expressions for \mathbf{v} and \dot{v}_{\parallel} are given by Eqs. (3.24) and (4.17) of Ref. [3]. Also, it is noted here that Eqs. (7) and (8) have similar structure as the

respective gyrokinetic equations of Refs. [10] and [11] [Eqs. (5.39) and (5.41) therein]. Since B_{\parallel}^* appears in the denominators of (7) and (8) a singularity occurs for $B_{\parallel}^* = 0$. For $\mathbf{E} = 0$ this singularity can be expressed by the critical parallel velocity $v_c = -\Omega/(\mathbf{b} \cdot \nabla \times \mathbf{b})$, where Ω is the gyrofrequency. Therefore, the theory is singular for large $|v_{\parallel}|$ at which \mathbf{v} and \dot{v}_{\parallel} diverge and consequently non-casual guiding center orbits occur and the guiding center conservation in phase space is violated [2]. It is the v_{\parallel} -dependence of \mathbf{A}^* [Eq. (2)] that produces the singularity. In order to regularize the singularity v_{\parallel} in (2) can be replaced by an antisymmetric function $g(z)$ with $z = v_{\parallel}/v_0$, where v_0 is some constant velocity [2]-[4]. The nonregularized theory presented here for simplicity is obtained for $g(z) = z$. In the regularized theory $g(z) \approx z$ should still hold for small $|z|$. For large $|z|$, however, g must stay finite such that with $v_0 \gg v_{\text{thermal}}$ one has $v_0 g(\infty) < v_c$. A possible choice for $g(z)$ is $g(z) = \tanh z$.

As in the Vlasov case [16] we consider the drift kinetic equation (1) for an axisymmetric equilibrium with $\mathbf{E} = 0$ in the vicinity of the magnetic axis. Since on axis the magnetic field becomes purely toroidal and dependent only on r one readily calculates

$$\nabla \times \mathbf{b} = \nabla \times \mathbf{e}_{\phi} = \frac{\mathbf{e}_z}{r},$$

$$B_{\parallel}^* = B, \quad \mathbf{B}^* = B\mathbf{e}_{\phi} + v_{\parallel} \frac{\mathbf{e}_z}{r}, \quad \nabla B(r) = \frac{dB}{dr} \mathbf{e}_r$$

and consequently

$$\mathbf{v} = \mathbf{v}_g = v_{\parallel} \mathbf{e}_{\phi} + \left(\frac{v_{\parallel}^2}{B} - \frac{\mu}{B} \frac{dB}{dr} \right) \mathbf{e}_z, \quad (10)$$

$$\dot{v}_{\parallel} = 0. \quad (11)$$

As expected on axis the guiding center velocity consists of a component parallel to \mathbf{B} and the curvature and grad- B drifts perpendicular to \mathbf{B} and parallel to the axis of symmetry. Also, the “acceleration” \dot{v}_{\parallel} vanishes because there is no parallel force and the drift kinetic equation (1) becomes

$$\mathbf{v} \cdot \frac{\partial f}{\partial \mathbf{R}} = 0. \quad (12)$$

Therefore, unlike the Vlasov description near axis the distribution function because of (11) can be any function of v_{\parallel} , which is a constant of motion, and

on axis can hold either $\nabla f \neq 0$ or $\nabla f = 0$. Thus, irrespective of the value of ∇f on axis one can either construct parallel currents or not by choosing f either symmetric or antisymmetric in v_{\parallel} . This is a significant difference from the Vlasov situation in which, if $\nabla f = 0$ on axis the obtainable particle distribution functions of the form $f(C_1, C_3)$ can not produce parallel currents. This discrepancy clearly relates to missing C_1 in the reduced phase space which results in a nontrivial loss of information for the particle motion. Note that near axis the overwhelming majority of the particles are passing and the parallel currents constructed in the framework of the drift kinetic theory may differ from the “actual” ones. In the context of the drift kinetic theory the two constants of motion, i.e. the energy $\mu B + 1/2 v_{\parallel}^2$ and the canonical momentum $r(A_{\phi} + v_{\parallel} b_{\phi})$ can be found from (1) everywhere by the method of characteristics. Thus, the complete set of constants of motions is obtained in the five dimensional phase space. These are recoverable on axis where v_{\parallel} , r , $A_{\phi}(r)$ (and μ) are conserved individually even if $\nabla f = 0$. Unlikely, in the respective Vlasov case the toroidal angular momentum constant of motion, C_2 , is missed when $\nabla f = 0$ is assumed. Also, for straight z -independent equilibria one has on axis $\nabla \times \mathbf{b} = \nabla \times \mathbf{e}_z = 0$, $\mathbf{B}^* = \mathbf{B} \mathbf{e}_z$, $\mathbf{v}_g = v_{\parallel} \mathbf{e}_z$, $\dot{v}_{\parallel} = 0$ and (1) is identically satisfied, implying that f can be any function of x, y, v_{\parallel} and μ . Therefore, unlike the Vlasov theory the dependence of f on v_{\parallel} is independent of toroidicity and therefore, regarding the formation of parallel currents, the drift kinetic theory can not distinguish equilibria of circular or straight magnetic axes.

Gyrokinetic theory

We will use the gyrokinetic equations of Ref. [7] which have been employed to a variety of applications (see for example the recent Refs. [12, 13, 14]). Eq. (1) remains identical in form where $f(\mathbf{R}, v_{\parallel}, \mu, t)$ is now the gyrocenter distribution function for ions. The gyrocenter velocity and “acceleration” are given by

$$\mathbf{v} = \mathbf{v}_g = v_{\parallel} \mathbf{b}_0 + \frac{B_0}{B_{0\parallel}^*} (\mathbf{v}_E + \mathbf{v}_{\nabla B} + \mathbf{v}_c), \quad (13)$$

$$\dot{v}_{\parallel} = -\frac{1}{v_{\parallel}} \mathbf{v}_g \cdot (\nabla \bar{\Phi} + \mu \nabla B_0). \quad (14)$$

Here, \mathbf{B}_0 is the equilibrium magnetic field, $\mathbf{b}_0 = \mathbf{B}_0/B_0$,

$$B_{0\parallel}^* = (\mathbf{B}_0 + v_{\parallel} \nabla \times \mathbf{b}_0) \cdot \mathbf{b}_0,$$

$\bar{\Phi}$ stands for the perturbed gyroaveraged electrostatic potential, and the $\mathbf{E} \times \mathbf{B}$ -drift velocity \mathbf{v}_E , the grad- B drift velocity $\mathbf{v}_{\nabla B}$, and the curvature drift velocity \mathbf{v}_c are given by

$$\mathbf{v}_E = -\frac{\nabla \bar{\Phi} \times \nabla B_0}{B_0^2}, \quad (15)$$

$$\mathbf{v}_{\nabla B} = \frac{\mu}{\Omega B_0} \mathbf{B}_0 \times \nabla B_0, \quad (16)$$

$$\mathbf{v}_c = \frac{\mu v_{\parallel}^2}{\Omega B_0^2} \mathbf{b}_0 \times \nabla \left(p_0 + \frac{B_0^2}{2} \right). \quad (17)$$

Note that as in the case of drift kinetic theory a similar singularity occurs at $B_{0\parallel}^* = 0$. In numerical applications this singularity was “avoided” by approximating $B_{0\parallel}^* = B_0$ (see for example Refs. [12, 13]). Consideration of the above equations for an axisymmetric equilibrium with $\mathbf{E} = 0$ near axis yields relations similar to (10), (11) and (12). Therefore, the above found discrepancies of the drift kinetic theory with the Vlasov one persist in the framework of the gyrokinetic theory. The structure of the reduced kinetic equations in conjunction with the symmetry of the equilibrium considered clearly indicate that this conclusion is independent of the particular drift kinetic or gyrokinetic equations.

In conclusion, first a singularity which occurs in both drift kinetic and gyrokinetic theories for large parallel particle velocities is usually eliminated in the literature by a rough approximation. Second, a comparison of the Vlasov equation with either the drift kinetic or the gyrokinetic equation near the magnetic axis of an axisymmetric equilibrium with vanishing electric field implies different properties of ∇f and, unlike Vlasov, non distinguishing of equilibria with straight and circular magnetic axes in connection with the formation of parallel currents. This relates to the loss of a Vlasov constant of motion in the reduced phase space. Consequently, different drift kinetic or gyrokinetic parallel currents may be created than the Vlasov ones. This indicates that the reduction of the phase space, even if made rigorously so that local conservation laws and Liouvillean invariance of the volume element is guaranteed, is associated with the loss of nontrivial physics which could put certain limits on the validity of the conclusions from drift kinetic or gyrokinetic simulations.

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